Algebraic Cycles

We first fix some notation. k is a field (possibly not alg. closed), and SmProj_k denotes the category of smooth projective varieties over k. A variety here is taken to be a separated reduced scheme of finite type. In particular we do not assume irreducible.

So now let X be in SmProj_k. The group of cycles $Z(X)$ is the free abelian group generated by irreducible subvarieties of X. An algebraic cycle is then a formal sum of such irreducible subvarieties, and if each summand has the same codimension i, we say the cycle has codimension i. The subgroup of all such cycles is denoted $Z^i(X)$. Clearly we have:

$$Z(X) = \oplus_i Z^i(X).$$

Ex:

1) $Z^1(X)$ is the group of divisors $\mathrm{Div}(X)$
2) $Z^{\dim X}(X)$ are finite formal sums of points. We can assign a degree to a point $p \in X$ by $[k(p):k] = \deg(p)$, which gives a $\mathbb{Z}$-linear function $Z^{\dim X}(X) \longrightarrow \mathbb{Z}$ by $\sum n_\alpha p_\alpha \longmapsto \sum n_\alpha \deg(p_\alpha)$. If $k = \bar{k}$, then $\deg(p_\alpha) = 1$ (See Prop. 1.21 in 3264 + All That).
3) If $Y \subset X$ is a subscheme, we can define an effective cycle in $Z(X)$. Denote by $Y_1, \dots, Y_s$ the irreducible components of Y, and $l_i = \mathrm{length}\, \mathcal{O}_{Y, Y_i}$. Then $\langle Y \rangle = \sum l_i Y_i$ is what we want.

Some things you can do:

- Products: $Z(X) \times Z(Y) \longrightarrow Z(X \times Y)$ by $(V, W) \longmapsto V \times W$. I do not think this is injective or surjective almost ever.

- Push forward: If $f: X \to Y$ is a proper morphism of k-varieties, and $Z \subset X$ is an irreducible subvariety, set $\deg(Z/f(Z))$ to be $[k(Z) : k(f(Z))]$ if $\dim Z = \dim f(Z)$ and 0 otherwise. Then the assignment $f_*(Z) = \deg(Z/f(Z))\, f(Z)$ extends linearly to a homomorphism $f_*: Z(X) \to Z(Y)$ (which is degree zero w.r.t. the grading).

- Intersection: If two subvarieties V and W in X intersect transversally along Z, then we can assign them an intersection multiplicity:

$$i(V \cdot W; Z) = \sum_{r=0}^{\dim X} (-1)^r \,\mathrm{length}_{\mathcal{O}_{V \cap W, Z}} \left(\mathrm{Tor}_r^{\mathcal{O}_{X,Z}} (\mathcal{O}_{V,Z}, \mathcal{O}_{W,Z}) \right).$$

  Then the intersection product is $V \cdot W = \sum_\alpha i(V \cdot W; Z_\alpha) Z_\alpha$, where Z_α are the subvarieties making up $V \cap W$. Note the higher Tor's are zero in the Cohen-Macaulay case, so the other terms are "correction terms".

- Pull back: Let $f: X \to Y$ be a morphism in SmProj_k and let $Z \subset Y$ be a subvariety. The graph Γ_f of f is a subvariety of $X \times Y$. If it meets $X \times Z$ transversally, we set $f^* Z = p_{X,*}(\Gamma_f \cdot (X \times Z))$. If f is flat, then $f^*(Z) = f^{-1}(Z)$, and this definition can be extended linearly to cycles.

- Correspondences: A correspondence from X to Y is a cycle in $X \times Y$. Given a correspondence A, it acts on cycles in X via:

$$A(T) = p_{Y,*}(A \cdot (T \times Y)) \in Z^{i+t-d}(Y),$$

  where $T \in Z^i(X)$, $A \in Z^t(X \times Y)$, $d = \dim X$. Note $t-d$ is called the degree of A.

Not all of these are always defined, especially if the cycles represent singular varieties. The idea is then to coarsen $Z(X)$ by some equivalence relation, so that by choosing representatives the above are always defined on equivalence classes.

<u>Def:</u> An equivalence relation $\sim$ on $Z(X)$ is called *adequate* if when restricted to $SmProj_k$ it satisfies
1) compatible with the grading and addition,
2) if $Z \sim 0$ on X, then $Z \times Y \sim 0$ in $Z(X \times Y)$ for all Y,
3) if $Z_1 \sim 0$ and $Z_1 \cdot Z_2$ is defined, then $Z_1 \cdot Z_2 \sim 0$,
4) if $Z \sim 0$ on $X \times Y$, then $p_{X,*}(Z) \sim 0$ on X
5) given $Z, W_1, \ldots, W_s \in Z(X)$, there is $Z' \sim Z$ such that $Z' \cdot W_i$ is defined for each i.

Having such an equivalence relation $\sim$, we set $C_\sim(X) = Z(X)/Z_\sim(X)$, where $Z_\sim(X)$ consists of cycles equivalent to zero. These are of course chosen so that the following lemma holds:

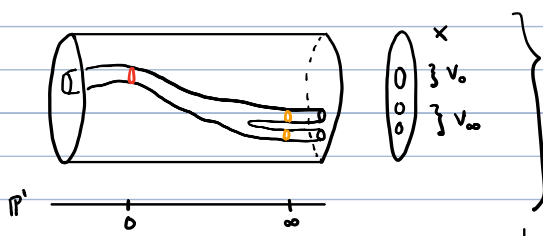
<u>Lemma:</u> For $\sim$ an adequate equivalence relation, $X \in SmProj_k$:
1) $C_\sim(X)$ is a ring under intersection,
2) for any $f: X \to Y$ in $SmProj_k$, the maps f^* & f_* induce homomorphisms $f_*: C_\sim(X) \to C_\sim(Y)$, and $f^*: C_\sim(Y) \to C_\sim(X)$, the latter a morphism of graded rings,
3) a correspondence of degree r induces $A_*: C^i_\sim(X) \to C^{i+d}_\sim(Y)$, and equivalent correspondences induce the same A_*.

We will now discuss various important equivalence relations.

Rational Equivalence

This is a generalization of the classical linear equivalence of divisors. Let $Y \subset X$ be an irreducible subvariety of codimension $i-1$. For a function $f \in K(Y)^\times$, then $\mathrm{div}(f)$ is a cycle of codimension i, and we say by definition $Z^i_{rat}(X)$ is generated by such cycles. Explicitly, a codimension i cycle $Z \sim_{rat} 0$ iff there is a finite collection of pairs (Y_α, f_α) such that $Z = \sum \mathrm{div}(f_\alpha)$.

An equivalent but perhaps more geometric definition is as follows. Two cycles V_0 and V_1 are said to be rationally equivalent if there is a cycle W on $\mathbb{P}^1 \times X$, not contained in any fiber $\{t\} \times X$, such that $W \cap (\{t_0\} \times X) - W \cap (\{t_1\} \times X) = A_0 - A_1$. Pictorally:



We then define the Chow groups as $CH^i(X) = Z^i(X)/Z^i_{rat}(X)$, $CH(X) = \bigoplus_i CH^i(X)$ $(= C_{rat}(X))$. While one can define this group for any variety, if we restrict to $SmProj_k$, it becomes an adequate equivalence relation.

Note: may only make sense in $SmProj_k$!

<u>Theorem:</u>
1) If $X \in SmProj_k$, then $CH(X)$ is a commutative graded ring under intersection product,
2) if $f: X \to Y$ is a morphism in $SmProj_k$, then f^* is a graded ring homomorphism, and f_* a graded group homomorphism of degree $\dim Y - \dim X$,
3) if $X, Y \in SmProj_k$, then $Z \in Corr^e_{rat}(X,Y)$ induces a group homomorphism of degree e,
4) if $i: Y \hookrightarrow X$ is a closed embedding and $j: X - Y = U \hookrightarrow X$, then we have an exact sequence
$$CH^q(Y) \xrightarrow{i_*} CH^q(X) \xrightarrow{j^*} CH^q(U) \longrightarrow 0.$$
5) The projection $p_X: X \times \mathbb{A}^n \to X$ induces an isomorphism $p_X^*: CH^i(X) \xrightarrow{\sim} CH^i(X \times \mathbb{A}^n)$.

Fulton actually proves this without property 5 (I guess he doesn't think its right?). His construction of the intersection product is actually slightly better.

Algebraic Equivalence

Supposing that X is smooth projective, we can replace $\mathbb{P}^1$ by any smooth irreducible curve C in the second definition of rational equivalence. Doing so, we arrive at algebraic equivalence

As an example of how this is coarser, note any rationally equivalent cycles are algebraically equivalent by taking $C = \mathbb{P}^1$. For the converse, if E is an elliptic curve and a, b are distinct points, then $Z = a - b \not\sim_{rat} 0$ as $g(E) = 1$. However, taking $C = E$ and $W = \Delta \subset E \times E$, we see that $Z \sim_{alg} 0$.

Smash Nilpotent Equivalence

Again $X \in \mathrm{SmProj}_k$. For a variety X and a cycle Z on X, we set $X^n = X \times \cdots \times X$ and $Z^n = Z \times \cdots \times Z$.

Def: $Z \sim_{\otimes} 0$ if and only if $Z^n \sim_{rat} 0$ on X^n for some positive integer n.

Prop: Smash-Nilpotent equivalence is an adequate equivalence relation. In particular $Z^i_{\otimes}(X) = \{ Z \in Z^i(X) \mid Z \sim_{\otimes} 0 \}$ is a subgroup of $Z^i(X)$.

The proof of this follows from the fact that rational equivalence is adequate. An important comparison result is the following:

Thm (Voisin - Voevodsky): $Z^i_{alg}(X)_{\mathbb{Q}} \subset Z^i_{\otimes}(X)_{\mathbb{Q}}$.

We might prove this later?

Homological Equivalence

Here, let F be a field of characteristic zero, and GrVect_F be the category of graded f.d. vector spaces over F.

Def: A Weil cohomology theory is a functor $H: \mathrm{SmProj}_k^{op} \longrightarrow \mathrm{GrVect}_F$ which satisfies:

1) There is a graded super-commutative cup product $\smile : H(X) \times H(X) \longrightarrow H(X)$,
2) Poincaré duality (trace iso: $\mathrm{Tr}: H^{2d}(X) \xrightarrow{\sim} F$, and perfect pairing $H^i \times H^{2d-i} \to H^{2d} \xrightarrow{\sim} F$),
3) Künneth formula holds: $H(X) \otimes H(Y) \xrightarrow{P_X^* \otimes P_Y^*} H(X \times Y)$ is a graded isomorphism,
4) Cycle maps: $cl_X : CH^i(X) \longrightarrow H^{2i}(X)$ which satisfy:
   - $f^* \circ cl_Y = cl_X \circ f^*$ and $f_* \circ cl_X = cl_Y \circ f_*$ for $f: X \to Y$ in SmProj_k
   - $cl_X(\alpha \cdot \beta) = cl_X(\alpha) \smile cl_X(\beta)$, where $\cdot$ is intersection product.
   - $\mathrm{Tr} \circ cl_p = \deg$ for points p. As notation, write $A^i(X) = \mathrm{Im}(cl_X) = H^{2i}_{alg}(X)$.
   - Weak Lefschetz: if $H \hookrightarrow X$ is a smooth hyperplane section, then $H^i(X) \longrightarrow H^i(H)$ is an isomorphism if $i < d-1$, and injective if $i = d-1$, and
   - Hard Lefschetz: $L(\alpha) = \alpha \smile cl_X(H)$ induces isomorphisms $L^{d-i}: H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X)$ for $0 \le i \le d$.

Some examples are of course:

1) If $\mathrm{char}(k)=0$ $k\subset\mathbb{C}$:
   - Betti Cohomology: $H^\bullet_B(X) = H^\bullet_{sing}(X)$
   - de Rham: $H^\bullet_{dR}(X_{an};\mathbb{C})$
   - algebraic de Rham: $H^\bullet_{dR}(X) = \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k})$
2) Étale cohomology. Recall:
$$H^\bullet_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell) = \varprojlim H^\bullet_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}/\ell^n\mathbb{Z})$$
$$H^\bullet_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_\ell) = H^\bullet_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell.$$
3) Crystalline cohomology (if k is perfect).

Def: Fix a Weil cohomology theory with cycle map cl_X. Then a cycle is homologically equivalent to zero, $Z\sim_{hom}0$ iff $cl_X(Z)=0$.

Note that a priori, this depends on the Weil cohomology theory.

Numerical Equivalence

Def: Let $X\in \mathrm{SmProj}_k$, for $Z\in Z^i(X)$ we say $Z\sim_{num}0$ if for every $W\in Z^{d-i}(X)$ such that $Z\cdot W$ is defined, $\deg(Z\cdot W)=0$.

We have the following relations

Lemma:
1) $Z^i_{alg}(X)\subset Z^i_{hom}(X)$,
2) $Z^i_{\otimes}(X)\subset Z^i_{hom}(X)$,
3) $Z^i_{hom}(X)\subset Z^i_{num}(X)$.

Proof:

(1): If $V_0 - V_1 = p_{X,*}(W\cdot(\{a\}-\{b\})\times X)$, then
$$cl_X(V_0-V_1) = p_{X,*}(cl_{X\times C}(W)\cup(cl_{X\times C}(a\times X) - cl_{X\times C}(b\times X))).$$

(2): Note if $Z^n\sim_{rat}0$, then $cl_{X^n}(Z^n) = cl_X(Z)\otimes\cdots\otimes cl_X(Z)\in H^{2in}(X^n)$. Since the cycle maps were defined on Chow groups, $cl_X(Z)=0$.

(3): If $cl_X(Z)=0$, then $\deg(Z\cdot W) = \mathrm{Tr}(cl_X(Z\cdot W)) = \mathrm{Tr}(cl_X(Z)\cup cl_X(W))=0$. ∎

Conjecture (Conjecture D(X)): If $k=\bar{k}$, then $Z^i_{hom}(X) = Z^i_{num}(X)$.

Known to be true for divisors ($i=1$) in all characteristics, in characteristic zero its also known in codimension 2, and is true for abelian varieties. We also know:

Theorem: If $k=\bar{k}$, then $Nm^i(X)_{\mathbb{Q}} = C^i_{num}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a f.d. vector space with dimension $\leq b_{2i}(X) = \dim_{\mathbb{Q}_\ell} H^{2i}_{\acute{e}t}(X)$.

Another important conjecture is:

Conjecture (Voevodsky): If $k = \bar{k}$, then $Z^i_{\otimes}(X) = Z^i_{num}(X)$.

Note while conjecture D would make homological equivalence independent of Weil cohomology theory, this conjecture is already independent and implies conjecture D for all Weil cohomology theories.

In conclusion then:

$$Z^i_{rat}(X) \subset Z^i_{alg}(X) \subset Z^i_{hom} \subseteq Z^i_{num} \subset Z^i(X)$$

over $\mathbb{Q}$:

$$Z^i_{alg}(X)_{\mathbb{Q}} \subset \underbrace{Z^i_{\otimes}(X)_{\mathbb{Q}} \subseteq Z^i_{hom}(X)_{\mathbb{Q}} \subseteq Z^i_{num}(X)_{\mathbb{Q}}}_{\text{expected equality in } k=\bar{k}.}$$

On the other hand, it's apparently provable that rational equivalence is the finest relation (Andres Book?).